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THE ASYMPTOTIC BEHAVIOR OF THE FIRST EIGENVALUE OF DIFFERENTIAL OPERATORS DEGENERATING ON THE BOUNDARY(1)

BY

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ABSTRACT. When L is a second order ordinary or elliptic differential operator, the principal eigenvalue for the Dirichlet problem and the corresponding principal (positive) eigenfunction u are known to exist and u is unique up to normalization. If further L has the form $\varepsilon\Sigma a_{ij}\partial^2/\partial x_i\partial x_j + \Sigma b_i\partial/\partial x_i$ then results are known regarding the behavior of the principal eigenvalue $\lambda = \lambda_\varepsilon$ as $\varepsilon\downarrow 0$. These results are very sharp in case the vector (b_i) has a unique asymptotically stable point in the domain ω where the eigenvalue problem is considered. In this paper the case where L is an ordinary differential operator degenerating on the boundary of ω is considered. Existence and uniqueness of a principal eigenvalue and eigenfunction are proved and results on the behavior of λ_ε as $\varepsilon\downarrow 0$ are established.

Introduction. The eigenvalue problem

$$\varepsilon x(1-x)\frac{d^2u}{dx^2} + \beta x(1-x)(x-\hat{x})\frac{du}{dx} = -\lambda u, \qquad 0 < x < 1, \beta < 0, \tag{1}$$
$$u(0) = u(1) = 0,$$

where $\hat{x}$ is a fixed point in (0,1), arises in a group of problems in genetics (see [6]). It is required to obtain asymptotic estimates for the minimum eigenvalue λ_ε as $\varepsilon\downarrow 0$.

If a second order ordinary differential operator is regular, i.e., if the leading coefficient of the differential operator does not vanish in the closed interval [0,1], then it is standard knowledge that the minimum eigenvalue exists, is positive and simple, and there is a corresponding eigenfunction (unique up to normalization) which is in $C^2[0,1]$ and is positive in (0,1). However, the leading coefficient of the operator in (1) degenerates at the boundary of [0,1] and it is not at all clear that a minimum eigenvalue exists, and if it does exist just what smoothness properties a corresponding eigenfunction has in [0,1].

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Disregarding these questions, G. F. Miller [6] gives an asymptotic estimate for λ_ε as $\varepsilon \to 0$ by techniques which may require further justification in order to satisfy the tenets of rigorous mathematics.

In this paper we shall give a complete analysis of a large class of eigenvalue problems which includes (1) as a special case. We shall show that if the order of degeneracy at the boundary of [0,1] of the leading coefficient of the differential operator is not too large, then all of the spectral properties for regular eigenvalue problems carry over. Further we shall give a rigorous derivation of the asymptotic estimate for λ_ε as $\varepsilon \to 0$.

The techniques developed in this paper carry over to the investigation of eigenvalue problems for a certain class of elliptic partial differential operators whose fundamental forms may degenerate at the boundary of the domain of definition. (See Remark at the end of §4.)

Finally we note that a paper of D. Ludwig [5] drew our attention to these problems, and we refer the reader to that paper for further examples and discussion. Other examples may be found in a paper of Fleming and Tsai [2].

1. Existence. Let

$$Lu \equiv a(x)u'' + b(x)u', \qquad x \in [0,1]. \tag{1.1}$$

As mentioned in the introduction, it is a well-known result of the theory of positive operators [4], that if the coefficients $a(x)$ and $b(x)$ are continuously differentiable and $a(x) > 0$ on [0,1], then there is a smallest positive number λ so that the eigenvalue problem

$$Lu = -\lambda u, \qquad x \in (0,1), \qquad u(0) = u(1) = 0, \tag{1.2}$$

has a solution u, which is positive in (0,1), and furthermore the eigenvalue λ is simple. This number λ is called the principal eigenvalue of the problem (1.2) and a corresponding positive eigenfunction, which is unique up to normalization, is called a principal eigenfunction.

In case the coefficient $a(x)$ degenerates at the boundary of [0, 1], the methods of [4] are no longer directly applicable. However, as we shall show, one may use the method of elliptic regularization to establish the existence of a principal pair (λ, u). The purpose of this section is to establish the existence of a likely candidate for a principal pair. Then in §2 we shall, by means of uniqueness results, show that the candidate which we have constructed is actually a principal pair for a degenerate eigenvalue problem of the form (1.2). We shall then bring this all together by means of formal statements in §3.

We shall make the following assumptions on the coefficients of the operator L in (1.1):

$$\begin{aligned}
&a,b \in C^1(0,1),\\
&a(x) > 0 \quad \text{in } (0,1),\\
&\frac{a(x)}{x^\alpha} \to a_0 > 0, \quad \frac{b(x)}{x^\beta} \to b_0 \quad \text{as } x \to 0,\\
&\frac{a(x)}{(1-x)^{\alpha'}} \to a_1 > 0, \quad \frac{b(x)}{(1-x)^{\beta'}} \to b_1 \quad \text{as } x \to 1,\\
&0 \leqslant \alpha,\alpha' < 2, \qquad \beta,\beta' \geqslant 0,\\
&\alpha < 1 + \beta, \qquad \alpha' < 1 + \beta'.
\end{aligned} \tag{1.3}$$

THEOREM 1.1. *Let the assumptions* (1.3) *hold. Then there exist a* $\lambda > 0$ *and a solution* $u(x)$ *of* (1.2) *such that* $u(x) > 0$ *in the open interval* (0,1) *and* $u(x)$ *is Hölder continuous with exponent* $\frac{1}{2}$ *in the closed interval* [0,1].

PROOF. We shall find it convenient here, as well as in the remainder of §1, to convert the eigenvalue problem (1.2) into an eigenvalue problem in a slightly different, but equivalent, form. Toward this end we introduce the function

$$p(x) = \exp \int_0^x \frac{b(t)}{a(t)}\, dt.$$

Since $\alpha < 1 + \beta$, $\alpha' < 1 + \beta'$, the last integral exists and is continuous in the closed interval [0,1]. Also

$$p'(x) = [b(x)/a(x)]p(x), \qquad x \in (0,1).$$

If we multiply the equation in (1.2) by p/a, we get the eigenvalue problem

$$(pu')' = -\lambda(p/a)u, \qquad x \in (0,1), \qquad u(0) = u(1) = 0. \tag{1.4}$$

Let $0 < \eta \leqslant 1$ and let

$$p_\eta(x) = \exp \int_0^x \frac{b(t)}{a(t) + \eta}\, dt.$$

Consider the eigenvalue problem

$$(p_\eta u')' = -\lambda[p_\eta/(a+\eta)]u, \qquad x \in (0,1), \qquad u(0) = u(1) = 0. \tag{1.5}$$

This is an eigenvalue problem for a nondegenerate elliptic operator; i.e.,

$$L_\eta u \equiv (a+\eta)u'' + bu' = -\lambda u, \qquad x \in (0,1), \qquad u(0) = u(1) = 0. \tag{1.6}$$

Therefore by [4] a principal pair (λ_η, u_η) exists and u_η is unique up to normalization. We normalize it by

$$\int_0^1 \frac{p_\eta u_\eta^2}{a+\eta} = 1. \tag{1.7}$$

If we replace u by u_η and λ by λ_η in (1.5) and then multiply by u_η we get, after integrating over [0,1],

$$\int_0^1 p_\eta (u_\eta')^2 = \lambda_\eta \int_0^1 \frac{p_\eta}{a+\eta} u_\eta^2 = \lambda_\eta. \tag{1.8}$$

In order to proceed with the proof we shall need

LEMMA 1.2. *λ_η is bounded for* $0 < \eta \leqslant 1$.

Let us assume, for the moment, that this is true and return to the proof of Theorem 1.1. In the sequel we shall use C as a generic positive constant, independent of η, which does not necessarily take the same value at each occurrence.

Using (1.7), (1.8) and Lemma 1.2, and the fact that $p_\eta \geqslant C$ we conclude that

$$\int_0^1 (u_\eta')^2 \leqslant C. \tag{1.9}$$

We also have

$$u_\eta(x) = \int_0^x u_\eta'(t)\, dt,$$

so that

$$u_\eta^2(x) \leqslant x \int_0^x (u_\eta')^2\, dt \leqslant Cx$$

by (1.9). Thus

$$u_\eta(x) \leqslant C\sqrt{x}\,. \tag{1.10}$$

Since $p_\eta(x) \leqslant C$, for all sufficiently small positive δ, independent of η,

$$\int_0^\delta \frac{p_\eta u_\eta^2}{a+\eta} \leqslant C \int_0^\delta x^{1-\alpha}\, dx \leqslant C\delta^{2-\alpha} < \frac{1}{4}\,. \tag{1.11}$$

Similarly we find

$$u_\eta(x) \leqslant C\sqrt{1-x}, \tag{1.12}$$

$$\int_{1-\delta}^1 \frac{p_\eta u_\eta^2}{a+\eta} < \frac{1}{4} \tag{1.13}$$

for all sufficiently small positive δ, independent of η. From (1.11) and (1.13) we get

$$\int_\delta^{1-\delta} \frac{p_\eta u_\eta^2}{a+\eta} > \frac{1}{2} \tag{1.14}$$

for all sufficiently small δ, independent of η.

From (1.9) and (1.10) and the fact that $L_\eta u_\eta = -\lambda_\eta u_\eta$ it follows that, for any $0 < \delta \leq \frac{1}{2}$,

$$\int_\delta^{1-\delta} |u_\eta''|^2 < C.$$

Thus the sets $\{u_\eta\}$ and $\{u_\eta'\}$, $0 < \eta \leq 1$, are equicontinuous and bounded in $[\delta, 1-\delta]$. Thus from Lemma 1.2 and the Arzela-Ascoli theorem there exists a sequence $\eta_m \to 0$ so that

$$\begin{aligned} &\lambda_{\eta_m} \to \lambda, \\ &u_{\eta_m}(x) \to u(x) \quad \text{uniformly, with the first two derivatives,} \\ &\qquad\qquad\qquad \text{in any compact subset of } (0,1). \end{aligned} \tag{1.15}$$

From (1.10) and (1.12) it is clear that u can be extended continuously to $[0,1]$ with $u(0) = u(1) = 0$ and indeed u is Hölder continuous in $[0,1]$ with exponent $\frac{1}{2}$. From these remarks and (1.15) it follows that u satisfies the boundary value problem (1.2).

Taking $\eta = \eta_m \to 0$ in (1.14) we get

$$\int_\delta^{1-\delta} \frac{pu^2}{a} \geq \frac{1}{2}. \tag{1.16}$$

Consequently, $u \not\equiv 0$. Further $u \geq 0$, since $u_\eta(x) > 0$ in $(0,1)$. From the latter fact about u, and $Lu = -\lambda u$, we get $Lu \leq 0$ in $[0,1]$. By (1.15) $u \in C^2(0,1)$ so that the strong maximum prinicple implies that either u takes its minimum only at $x = 0$ or $x = 1$, or else $u \equiv 0$. But as already pointed out the latter cannot occur so that $u(x) > 0$ for $x \in (0,1)$.

Finally $\lambda > 0$. Indeed, in the contrary case $\lambda = 0$ so that $Lu = 0$ in $(0,1)$. But then the maximum principle implies $u \equiv 0$, which is a contradiction.

The proof of Theorem 1.1 will be complete as soon as we have given the

PROOF OF LEMMA 1.2. By the comparison Lemma 2.2 of [1], or by a standard argument involving the characterization of the minimum eigenvalue of a positive selfadjoint operator on a Hilbert space, we have that $\lambda_\eta \leq \mu_\eta$, where μ_η is the principal eigenvalue for the Dirichlet problem for the elliptic operator L_η in any smaller interval, say $[1/3, 2/3]$. In this interval L_η (for $0 \leq \eta \leq 1$) is uniformly elliptic (both in x and η) and hence by the continuity Lemma 3.1 of [1], $\mu_\eta \to \mu_{\eta_0}$ as $\eta \to \eta_0$. Hence μ_η is continuous for $0 \leq \eta \leq 1$ and so is bounded. Since μ_η dominates λ_η we have completed the proof.

REMARK. If $\alpha \leq \frac{3}{2}$ one can show that for every $0 < \theta < 1$,

$$u(x) \leq C_\theta x^\theta. \tag{1.17}$$

Indeed, for $x \in [0,\delta]$ take $w = Ax^\theta$, where $A\delta^\theta = C\delta^{1/2}$, C being taken as in (1.10). Then as is easily seen

(1.18) $$(p_\eta w')' \leqslant -\lambda_\eta[p_\eta/(a+\eta)]u_\eta, \qquad x \in [0,\delta],$$

provided $Ax^{\theta-2} \geqslant Kx^{1/2-\alpha}$ in $[0,\delta]$ for some suitable constant K. This can be accomplished provided δ is sufficiently small since $\frac{5}{2}-\theta-\alpha>0$. δ and K can be chosen independent of η. Since (1.18) holds and $w \geqslant u_\eta$ at $x=0$ and $x=\delta$, it follows from the maximum principle that $u_\eta(x) \leqslant w(x)$ in $[0,\delta]$. Letting $\eta \to 0$ gives our assertion.

2. **Uniqueness.** In the following (λ,u) is the pair whose existence is asserted in Theorem 1.1

THEOREM 2.1. *Suppose $v \in C[0,1] \cap C^2(0,1)$, v real, pv^2/a is integrable over* $[0,1]$, *and for μ real*

(2.1) $$Lv = -\mu v, \qquad x \in (0,1), \qquad v(0)=v(1)=0.$$

Then $\mu \geqslant \lambda$.

Before we begin the proof it will be convenient to establish

LEMMA 2.2. *Under the hypotheses of Theorem* 2.1

(2.2) $$\int_0^1 p(v')^2 \leqslant \mu\int_0^1 \frac{p}{a}v^2.$$

PROOF. Suppose x_1 and x_2 in (0,1) are consecutive zeros of v. Multiplying the equation in (2.1) by pv/a, integrating over $[x_1,x_2]$, and then integrating by parts gives

$$\int_{x_1}^{x_2} p(v')^2 = \mu\int_{x_1}^{x_2}\frac{pv^2}{a}.$$

Hence, summing over the intervals determined by the consecutive zeros would give (2.2) provided we could integrate by parts when $x_1=0$ or $x_2=1$.

Let us consider the case where $x_1=0$, $x_2=1$, and $v>0$ in (0,1). Multiplying the equation in (2.1) by pv/a, integrating over $[y_1,y_2]$ $(0<y_1<y_2<1)$ and then integrating by parts gives

$$\int_{y_1}^{y_2} p(v')^2 = p(y_2)v'(y_2)v(y_2) - p(y_1)v'(y_1)v(y_1) + \mu\int_{y_1}^{y_2}\frac{pv^2}{a}.$$

Since $v>0$ in (0, 1) and vanishes at the endpoints, there exist sequences $y_{1n}\downarrow 0$ and $y_{2n}\uparrow 1$ so that $v'(y_{1n}) \geqslant 0$ and $v'(y_{2n}) \leqslant 0$. Using this in the above equality gives (2.2).

Since the cases where $x_1=0$, $x_2<1$ or $x_2>0$, $x_2=1$ can be handled in a similar way, we have concluded the proof.

PROOF OF THEOREM 2.1. From (1.2) and (2.1) we have

(2.3) $$(pu')' = -\lambda(p/a)u,$$

$$(2.4) \qquad (pv')' = -\mu(p/a)v.$$

Let x_1 and x_2 be consecutive zeros of v. Without loss of generality we may suppose $v > 0$ in (x_1,x_2). Multiply (2.3) by v, subtract it from (2.4) multiplied by u and integrate over $[x_1,x_2]$. If $0 < x_1 < x_2 < 1$ we may integrate by parts to get

$$p(x_2)v'(x_2)u(x_2) - p(x_1)v'(x_1)u(x_1) = (\lambda - \mu)\int_{x_1}^{x_2} \frac{p}{a} uv.$$

But since $v > 0$ in (x_1,x_2), $v'(x_2) \leqslant 0$ and $v'(x_1) \geqslant 0$, so that

$$(\lambda - \mu)\int_{x_1}^{x_2} \frac{p}{a} uv \leqslant 0.$$

If we suppose $\mu < \lambda$ we get a contradiction since $puv/a > 0$ in (x_1,x_2).

In case $x_1 = 0$ or $x_2 = 1$ we must be somewhat more careful since u' and v' may become unbounded at the endpoints. Suppose $x_1 = 0$ and $x_2 = 1$. Then for $0 < y_1 < y_2 < 1$ we have

$$[pv'u - pu'v]_{y_1}^{y_2} = (\lambda - \mu)\int_{y_1}^{y_2} \frac{p}{a} uv.$$

Integrate with respect to y_1 over the interval $(0,\delta)$ and then divide by δ to get

$$(2.5) \qquad \begin{aligned} &[pv'u - pu'v](y_2) - \frac{1}{\delta}\int_0^{\delta} [pv'u - pu'v](y_1)\, dy_1 \\ &\qquad = (\lambda - \mu)\frac{1}{\delta}\int_0^{\delta}\int_{y_1}^{y_2} \frac{p}{a} uv\, dx\, dy_1. \end{aligned}$$

Now,

$$\frac{1}{\delta}\left|\int_0^{\delta} pu'v\, dy_1\right| \leqslant \frac{C}{\delta}\left[\int_0^{\delta}(u')^2\, dy_1\right]^{1/2}\left[\int_0^{\delta} v^2\, dy_1\right]^{1/2}.$$

Using Lemma 2.2 it follows in exactly the same way as the proof of (1.10) that $v(x) = O(\sqrt{x})$ as $x \to 0$. Thus it follows that $\int_0^{\delta} v^2\, dy_1 = O(\delta^2)$ as $\delta \to 0$. Since u' is square integrable on $[0,1]$ we get $\int_0^{\delta}(u')^2 = o(1)$ as $\delta \to 0$. Thus we find

$$\frac{1}{\delta}\left|\int_0^{\delta} pu'v\right| \to 0 \quad \text{as } \delta \to 0.$$

Using exactly the same reasoning we also have

$$\frac{1}{\delta}\left|\int_0^{\delta} pv'u\right| \to 0 \quad \text{as } \delta \to 0.$$

Returning to (2.5) and noting that $\int_{y_1}^{y_2}(p/a)uv$ is continuous at $y_1 = 0$ we find, upon letting $\delta \to 0$, that

$$[pv'u - pu'v](y_2) = (\lambda - \mu)\int_0^1 \frac{p}{a}uv.$$

Integrating with respect to y_2 over $[\delta, 1-\delta]$, dividing by δ, and then letting $\delta \to 0$ we obtain, by the same argument as before,

$$0 = (\lambda - \mu)\int_0^1 \frac{p}{a}uv,$$

which is impossible if $\lambda > \mu$.

If $x_1 = 0$ and $x_2 < 1$, or $x_1 > 0$ and $x_2 = 1$, then similar reasoning shows that

$$(\lambda - \mu)\int_0^{x_2} \frac{p}{a}uv \leqslant 0 \quad \text{or} \quad (\lambda - \mu)\int_{x_1}^1 \frac{p}{a}uv \leqslant 0,$$

which is a contradiction if $\lambda > \mu$.

DEFINITION 2.3. *A regular eigensolution of* (2.1) *is a solution which satisfies the hypotheses of Theorem* 2.1. *The corresponding eigenvalue is called a regular eigenvalue.*

REMARK. Theorem 2.1 asserts that λ is the smallest real regular eigenvalue. If $\alpha, \alpha' < \frac{3}{2}$ then we can assert more; namely that λ is the smallest real eigenvalue for the problem (2.1) where the eigenfunctions that are allowed into the competition are only assumed to be in $C[0,1] \cap C^2(0,1)$. To see this we suppose for definiteness that $v > 0$ in $(0,1)$. Let $\eta > 0$ and let x_η and y_η be zeros of $w_\eta = v - \eta$ so that $w_\eta > 0$ in (x_η, y_η) and $x_\eta \downarrow 0$, $y_\eta \uparrow 1$ as $\eta \to 0$. As before we have

$$(pw_\eta')' = -\mu(p/a)w_\eta - \mu\eta(p/a), \quad (pu')' = -\lambda(p/a)u.$$

Multiply the second equation by w_η and subtract it from the first equation multiplied by u and then integrate over $[x_\eta, y_\eta]$ to get

$$[pw_\eta' u]_{x_\eta}^{y_\eta} = (\lambda - \mu)\int_{x_\eta}^{y_\eta} \frac{p}{a} w_\eta u - \mu\eta\int_{x_\eta}^{y_\eta} \frac{p}{a} u.$$

The first integral on the right increases as $\eta \to 0$ and, since $\alpha, \alpha' < \frac{3}{2}$, by the remark at the end of §1, $u(x) = O(x^\theta(1-x)^\theta), 0 < \theta < 1$, so that the second term on the right goes to zero as $\eta \to 0$. Thus for all sufficiently small η the right-hand side is positive if $\lambda > \mu$. But since $w_\eta'(x_\eta) \geqslant 0$ and $w_\eta'(y_\eta) \leqslant 0$, the left-hand side is $\leqslant 0$, which is a contradiction.

THEOREM 2.4. *In addition to the assumptions* (1.3) *assume that* $(x^{-\beta}b(x))'$, $(x^{-\alpha}a(x))'$ *and* $(x^{-\alpha}a(x))''$ *are bounded in* $(0,c]$, $0 < c < 1$. *Then all of the real eigenvalues of the boundary value problem* (2.1) *are simple.*

PROOF. Suppose μ is a real eigenvalue and v_1 and v_2 are real linearly independent eigenfunctions corresponding to μ. For every $\delta > 0$ there exists a linear combination $v = c_1v_1 + c_2v_2$ such that $v(\delta) = 0$, and $v \not\equiv 0$ in $[0,\delta]$.

Thus v is a nontrivial solution to the boundary value problem

$$Lv = -\mu v, \qquad x \in (0,\delta), \qquad v(0) = v(\delta) = 0. \tag{2.6}$$

We may assume that max $v(x) > 0$ in $[0,\delta]$. Then for every sufficiently small positive η there exist numbers $0 < x_\eta < y_\eta < \delta$ such that if $w_\eta = v - \eta$, then $w_\eta(x) > 0$ if $x_\eta < x < y_\eta$, and $w_\eta(x_\eta) = w_\eta(y_\eta) = 0$. Further $x_\eta \downarrow x_0 > 0$ and $y_\eta \uparrow y_0 \leqslant \delta$ as $\eta \downarrow 0$.

We have

$$aw_\eta'' + bw_\eta' = -\mu w_\eta - \mu\eta.$$

Multiplying both sides by $x^{-\gamma}w_\eta$ and integrating over $[x_\eta, y_\eta]$ we get

$$\int_{x_\eta}^{y_\eta} x^{-\gamma}aw_\eta'' w_\eta + \int_{x_\eta}^{y_\eta} x^{-\gamma}bw_\eta' w = -\mu\int_{x_\eta}^{y_\eta} x^{-\gamma}w_\eta^2 - \mu\eta\int_{x_\eta}^{y_\eta} x^{-\gamma}w_\eta. \tag{2.7}$$

We shall take γ so that

$$\gamma < \alpha < \gamma + 1 < 2, \qquad \gamma \neq \beta - 2. \tag{2.8}$$

For notational convenience let us set $\tilde{a}(x) = x^{-\alpha}a(x)$, $\tilde{b}(x) = x^{-\beta}b(x)$; then

$$\begin{aligned}\int_{x_\eta}^{y_\eta} x^{-\gamma}aw_\eta'' w_\eta &= -\int_{x_\eta}^{y_\eta}(x^{\alpha-\gamma}\tilde{a}w_\eta)'w_\eta' \\ &= -\int_{x_\eta}^{y_\eta} x^{\alpha-\gamma}\tilde{a}\,(w_\eta')^2 - \int_{x_\eta}^{y_\eta}(x^{\alpha-\gamma}\tilde{a})'w_\eta w_\eta' \\ &= -\int_{x_\eta}^{y_\eta} x^{-\gamma}a(w_\eta')^2 + \frac{1}{2}\int_{x_\eta}^{y_\eta}(x^{\alpha-\gamma}\tilde{a})''w_\eta^2.\end{aligned}$$

Also

$$\int_{x_\eta}^{y_\eta} x^{-\gamma}bw_\eta' w_\eta = -\frac{1}{2}\int_{x_\eta}^{y_\eta}(x^{\beta-\gamma}\tilde{b})'w_\eta^2.$$

Substituting into (2.7) gives

$$\begin{aligned}\int_{x_\eta}^{y_\eta} x^{-\gamma}a(w_\eta')^2 &= \frac{1}{2}\int_{x_\eta}^{y_\eta}(x^{\alpha-\gamma}\tilde{a})''w_\eta^2 - \frac{1}{2}\int_{x_\eta}^{y_\eta}(x^{\beta-\gamma}\tilde{b})'w_\eta^2 \\ &\quad + \mu\int_{x_\eta}^{y_\eta} x^{-\gamma}w_\eta^2 + \mu\eta\int_{x_\eta}^{y_\eta} x^{-\gamma}w_\eta.\end{aligned} \tag{2.9}$$

Since $\alpha - \gamma > 0$ and $\alpha - \gamma - 1 < 0$ we have

$$\begin{aligned}(x^{\alpha-\gamma}\tilde{a})'' &= (\alpha-\gamma)(\alpha-\gamma-1)x^{\alpha-\gamma-2}\left[\tilde{a}(x) + O(x)\right] \\ &\leqslant (\alpha-\gamma)(\alpha-\gamma-1)x^{\alpha-\gamma-2}\left[a_0 + O(x)\right] < 0\end{aligned} \tag{2.10}$$

in $(0,\delta)$ provided δ is sufficiently small.

We shall need to use the inequality

$$\int_{x_\eta}^{y_\eta} x^{p-2}w_\eta^2 \leqslant \frac{4}{(p-1)^2}\int_{x_\eta}^{y_\eta} x^p(w_\eta')^2, \qquad p \neq 1. \tag{2.11}$$

This is an immediate consequence of the equality

$$\int_{x_\eta}^{y_\eta} x^{p-2} w_\eta^2 = -\frac{2}{(p-1)} \int_{x_\eta}^{y_\eta} x^{p-1} w_\eta w_\eta'$$

and the Cauchy-Schwarz inequality.

Returning to the proof of Theorem 2.4 and noting that $(x^{\beta-\gamma}\tilde{b})' = O(x^{\beta-\gamma-1})$ and $\beta - \gamma - 1 \neq 1$ we get from (2.11) that

$$\int_{x_\eta}^{y_\eta} (x^{\beta-\gamma}\tilde{b})' w_\eta^2 \leq C \int_{x_\eta}^{y_\eta} x^{\beta-\gamma+1} (w_\eta')^2. \tag{2.12}$$

Also since $\gamma < 1$, we may apply (2.11) again to get

$$\int_{x_\eta}^{y_\eta} x^{-\gamma} w_\eta^2 \leq C \int_{x_\eta}^{y_\eta} x^{-\gamma+2} (w_\eta')^2. \tag{2.13}$$

Using (2.10), (2.12) and (2.13) in (2.9) we get, provided δ is sufficiently small,

$$\frac{1}{2} a_0 \int_{x_\eta}^{y_\eta} x^{\alpha-\gamma} (w_\eta')^2 \leq C \int_{x_\eta}^{y_\eta} (x^{\beta-\gamma+1} + x^{2-\gamma})(w_\eta')^2 + \mu\eta \int_{x_\eta}^{y_\eta} x^{-\gamma} w_\eta.$$

Noting that $w_\eta' = v'$ and $\gamma < 1$, if we allow $\eta \to 0$ we get

$$\frac{1}{2} a_0 \int_{x_0}^{y_0} x^{\alpha-\gamma} (v')^2 \leq C \int_{x_0}^{y_0} (x^{\beta-\gamma+1} + x^{2-\gamma})(v')^2.$$

But since $\alpha < \beta + 1$ and $\alpha < 2$, this is clearly a contradiction if δ is sufficiently small. The proof is complete.

3. **The principal eigenvalue.** In this section we shall assume, in addition to (1.3), the conditions

(3.1) $(x^{-\beta}b(x))'$, $(x^{-\alpha}a(x))'$, and $(x^{-\alpha}a(x))''$ *are bounded functions in some interval* $(0,c]$.

Recall that in §1 we called the smallest number λ so that the eigenvalue problem (1.2) has a solution the principal eigenvalue, and a corresponding eigenfunction u, which is positive in (0,1), a principal eigenfunction. In Definition 2.3 we defined a regular eigenfunction and eigenvalue.

THEOREM 3.1. *Let* (1.3) *and* (3.1) *hold. Then there exists a unique regular principal pair* (λ,u) *so that* $\int_0^1 pu^2/a = 1$, *and u is Hölder continuous on* [0,1] *with exponent* $\frac{1}{2}$.

PROOF. Theorem 1.1 establishes the existence of a candidate (λ,u). If $(\hat{\lambda},\hat{u})$ is another regular eigenpair, $\hat{\lambda}$ real, then Theorem 2.1 shows that $\hat{\lambda} \geq \lambda$. Thus λ is the smallest real regular eigenvalue for the problem (1.2). If, in addition, $\hat{u} > 0$ in (0,1), then the same theorem shows that $\lambda \geq \hat{\lambda}$, so that $\lambda = \hat{\lambda}$. Further, if $\int_0^1 p\hat{u}^2/a = 1$, then Theorem 2.4 shows that $u = \hat{u}$.

In the proof of Theorem 1.1 we showed that there exists a sequence $\{\eta_m\}$ so that $\lambda_{\eta_m} \to \lambda$, $u_{\eta_m} \to u$. Actually the same method shows that from any sequence $\{\tilde{\eta}_m\}$ one may extract a subsequence $\{\eta'_m\}$ so that $\lambda_{\eta'_m} \to \tilde{\lambda}$ and $u_{\eta'_m} \to \tilde{u} > 0$ in (0,1). If the $u_{\eta'_m}$ are normalized by $\int_0^1 p_{\eta'_m} u^2_{\eta'_m}/a = 1$, then by uniqueness $\lambda = \tilde{\lambda}$, $u = \tilde{u}$. Thus we have proved

COROLLARY 3.2. *Under the assumptions* (1.3), (3.1) *and* $\int_0^1 p_\eta u_\eta^2/(a + \eta) = 1$, *as* $\eta \to 0$,

(3.2) $$\lambda_\eta \to \lambda, \qquad u_\eta(x) \to u(x), \qquad x \in [0,1].$$

Let $0 \leq x_1 < x_2 \leq 1$ and consider the eigenvalue problem

(3.3) $$Lu = -\mu u, \qquad x \in (x_1,x_2), \qquad u(x_1) = u(x_2) = 0.$$

Denote the regular principal eigenvalue by $\lambda(x_1,x_2)$. Denote by $\lambda_\eta(x_1,x_2)$ the principal eigenvalue corresponding to L_η in $[x_1,x_2]$. By the comparison Lemma 2.2 of [1] we have $\lambda_\eta \leq \lambda_\eta(x_1,x_2)$. Let $\eta \to 0$ and we have proved

COROLLARY 3.3. *Under the assumptions* (1.3) *and* (3.1),

(3.4) $$\lambda \leq \lambda(x_1,x_2),$$

where $\lambda(x_1,x_2)$ *is the regular principal eigenvalue for L with zero Dirichlet data on the boundary of* $[x_1,x_2] \subseteq [0,1]$.

In order to obtain the asymptotic estimates of the next section it is necessary to get a certain explicit form for the regular principal eigenvalue λ in terms of u and u'. This is the content of our next theorem.

THEOREM 3.4. *Under the assumptions* (1.3) *and* (3.1),

(3.5) $$\lambda = \int_0^1 p(u')^2 \Big/ \int_0^1 \frac{p}{a} u^2$$

where u is any regular eigenfunction corresponding to the regular principal eigenvalue λ.

PROOF. In Lemma 2.2 we showed that

(3.6) $$\int_0^1 p(u')^2 \leq \lambda \int_0^1 \frac{p}{a} u^2.$$

Let $\delta(x)$ be a continuous positive function on (0,1) so that $\delta(x) \to 0$ as $x \to 0$ and as $x \to 1$, and

$$\int_0^1 \frac{\delta^2(x)}{x(1-x)}\, dx = \infty.$$

Then there exist sequences $x_n \to 0, y_n \to 1$ so that

(3.7) $$|u'(x_n)| \leq \delta(x_n)/\sqrt{x_n}\,, \quad |u'(y_n)| \leq \delta(y_n)/\sqrt{1-y_n}\,.$$

For in the contrary case the finiteness of the left-hand integral in (3.6) would be violated. If we multiply $Lu = -\lambda u$ by u and integrate over $[x_n, y_n]$ we get

$$\int_{x_n}^{y_n} p(u')^2 = p(y_n)u'(y_n)u(y_n) - p(x_n)u'(x_n)u(x_n) + \lambda \int_{x_n}^{y_n} \frac{p}{a} u^2.$$

Letting $x_n \to 0$, $y_n \to 1$, and taking account of the inequalities (1.10), (1.12) and (3.7), we see that (3.5) is established.

As a final theorem in this section we shall give more information on the operator determined by L acting on certain functions having zero Dirichlet data on the boundary of $[0,1]$. Toward this end let $L^2(p/a)$ be the space of (equivalence classes of) functions square integrable with respect to the measure $(p/a)dx$ on $[0,1]$. Let $\mathfrak{D}$ be the collection of elements u in $L^2(p/a)$ with the additional properties:

(i) $u \in C[0,1] \cap C^1(0,1)$;

(ii) $u(0) = u(1) = 0$;

(iii) u' is locally absolutely continuous in $(0,1)$;

(iv) $Lu \in L^2(p/a)$.

Strictly speaking the elements of $\mathfrak{D}$ are equivalence classes each of which contains a function satisfying the properties (i) through (iv). However, we shall adopt the usual colloquial device of treating the elements of $\mathfrak{D}$ as functions.

THEOREM 3.5. *The operator $M = L|\mathfrak{D}$ is selfadjoint in $L^2(p/a)$ and has a completely continuous inverse.*

PROOF. Let $\eta > 0$ and let $\mathfrak{D}_\eta$ be defined as above with p/a replaced by $p_\eta/(a+\eta)$. Let $f \in L^2(p/a)$ and let u_η be a solution in $\mathfrak{D}_\eta$ to $L_\eta u_\eta = f$, or, what is the same thing, a solution to

$$(p_\eta u_\eta')' = p_\eta f/(a+\eta). \tag{3.8}$$

By the standard theory of elliptic ordinary differential operators such a solution exists and is unique. Since the coefficients of L_η are real, there is no loss in generality in assuming that f is real, so that the solution u_η is also real.

Using exactly the same technique as employed in the proof of Lemma 2.2 we may prove that for every $\varepsilon > 0$,

$$\int_0^1 p_\eta (u_\eta')^2 < \frac{1}{\varepsilon}\int_0^1 \frac{p}{a} f^2 + \varepsilon \int_0^1 \frac{p_\eta}{a+\eta} u_\eta^2. \tag{3.9}$$

Arguing as in the proof of Lemma 1.2 we find that

$$\frac{u_\eta(x)^2}{a+\eta} < \frac{x}{a}\int_0^1 (u_\eta')^2 \quad \text{and} \quad \frac{u_\eta(x)^2}{a+\eta} < \frac{1-x}{a}\int_0^1 (u_\eta')^2. \tag{3.10}$$

Thus,

$$(3.11)\qquad \int_0^1 \frac{u_\eta^2}{a+\eta} < \left\{ \int_0^{1/2} \frac{x}{a} + \int_{1/2}^1 \frac{1-x}{a} \right\} \int_0^1 (u_\eta')^2 = C\int_0^1 (u_\eta')^2.$$

If we use this inequality in (3.9), noting that $p_\eta > C > 0$ independent of η, $0 < \eta < 1$, we get, upon choosing ε sufficiently small, but fixed,

$$(3.12)\qquad \int_0^1 \frac{u_\eta^2}{a+\eta} + \int_0^1 (u_\eta')^2 < C\int_0^1 \frac{p}{a} f^2.$$

From the fact that

$$(3.13)\qquad (a+\eta)u_\eta'' = f - bu_\eta',$$

and from (3.12) we find that for every $\delta > 0$ there exists a constant $C_\delta > 0$ so that

$$(3.14)\qquad \int_\delta^{1-\delta} (u_\eta'')^2 < C_\delta \int_0^1 \frac{p}{a} f^2.$$

It follows from (3.12) and (3.14) that $\{u_\eta\}$ and $\{u_\eta'\}$ for $0 < \eta < 1$ are bounded equicontinuous sets of functions on $[\delta, 1-\delta]$. Thus by the Arzela-Ascoli theorem there exists a sequence $\{\eta_k\}$ so that

$$(3.15)\qquad \begin{cases} u_{\eta_k} \to u \\ \qquad\qquad \text{uniformly in } [\delta, 1-\delta]. \\ u_{\eta_k}' \to u' \end{cases}$$

From (3.13) and (3.15) it follows that

$$(3.16)\qquad u_{\eta_k}'' \to u'' \text{ uniformly in } [\delta, 1-\delta],$$

and, moreover,

$$(3.17)\qquad Lu = f.$$

It follows from this, (3.10), (3.12) and (3.15) that $u \in \mathfrak{D}$. If $v \in \mathfrak{D}$ and $Lv = 0$, it follows that $v \in C^2(0,1)$ so that by the maximum principle $v = 0$. Thus the solution in $\mathfrak{D}$ to (3.17) is unique.

The inverse map to (3.17) given by $Rf = u$ is well defined from $L^2(p/a) \to \mathfrak{D}$. From (3.12), (3.15) and Fatou's lemma it follows that R is bounded. Actually R is the inverse of $M = L|\mathfrak{D}$. This is immediate since by the very definition of R and $\mathfrak{D}$, R maps onto $\mathfrak{D}$ in a one-to-one manner.

The operator R is selfadjoint in $L^2(p/a)$. To see this we first note that $C_0^\infty(0,1)$ is dense in $L^2(p/a)$, and secondly, $R_\eta = M_\eta^{-1}$ is selfadjoint in $L^2(p_\eta/(a+\eta))$ where $M_\eta = L_\eta|\mathfrak{D}_\eta$. Thus if $f,g \in C_0^\infty(0,1)$ we have by (3.15)

$$\int_0^1 Rf\,\bar{g}\,\frac{p}{a} = \lim_{\eta\to 0} \int_0^1 R_\eta f \bar{g}\,\frac{p_\eta}{a+\eta} = \lim_{\eta\to 0}\int_0^1 f R_\eta \bar{g}\,\frac{p_\eta}{a+\eta} = \int_0^1 f\,\overline{Rg}\,\frac{p}{a}.$$

Since R is symmetric on a dense set, and bounded, it is selfadjoint. But

$M = R^{-1}$ so that M is selfadjoint in $L^2(p/a)$.

It remains to prove that R is compact. Let $\{f_m\}$ be a bounded sequence in $L^2(p/a)$ and $u_m = Rf_m$. From (3.10) and (3.12) with u_η replaced by u_m and $a + \eta$ replaced by a we have

$$\int_0^\delta u_m^2 \frac{p}{a} + \int_{1-\delta}^1 u_m^2 \frac{p}{a} = o(1) \quad \text{as } \delta \to 0 \text{ independently of } m. \tag{3.18}$$

Rellich's compactness theorem taken in conjunction with (3.18) and (3.12), with u_η replaced by u_m, $a + \eta$ replaced by a, and f replaced by f_m, implies there exists a subsequence of $\{u_m\}$ which converges in $L^2(p/a)$. The proof is complete.

4. Asymptotic behavior. In this section we shall consider the elliptic operator

$$L_\varepsilon u \equiv (\varepsilon/2)au'' + bu' \qquad (\varepsilon > 0). \tag{4.1}$$

We shall denote by λ_ε the regular principal eigenvalue for this operator with zero Dirichlet data at the boundary of [0,1]. Our object is to describe the asymptotic behavior of λ_ε as $\varepsilon \to 0$ for a class of operators which include those mentioned in the introduction.

In addition to the assumptions (1.3) and (3.1) on $a(x)$ and $b(x)$ we shall require the additional assumptions:

$$\begin{gathered} b_0 > 0, \quad b_1 < 0; \\ \textit{there exists an } \hat{x} \in (0,1) \textit{ such that } b(x) > 0 \\ \textit{if } x \in (0,\hat{x}) \textit{ and } b(x) < 0 \textit{ if } x \in (\hat{x},1). \end{gathered} \tag{4.2}$$

Under these assumptions it is easily seen that all solutions $x = x(t)$ of $dx/dt = b(x)$, $0 < x(0) < 1$, remain in (0,1) and converge to $\hat{x}$ as $t \to \infty$.

For any absolutely continuous function $\phi(t)$, $0 \leqslant t \leqslant T$, define

$$I_T(\phi) = \int_0^T \frac{\left[\phi'(t) - b(\phi(t))\right]^2}{a(\phi(t))}\, dt. \tag{4.3}$$

Let $\Phi(x,T;\alpha,\beta)$ be the class of absolutely continuous functions defined on $[0,T]$ such that $\phi(0) = x \in (\alpha,\beta)$, $\phi(t) \in (\alpha,\beta)$ for all $t \in [0,T)$ and $\phi(T) = \alpha$ or $\phi(T) = \beta$. For any $0 < \delta < \frac{1}{2}$ let

$$\begin{aligned} J_\delta &= \inf\{I_T(\phi): \phi \in \Phi(\hat{x},T;\delta,1-\delta)\}, \\ J &= \inf\{I_T(\phi): \phi \in \Phi(\hat{x},T;0,1)\}. \end{aligned} \tag{4.4}$$

Notice that both ϕ and T are arbitrary.

In addition to the restrictions (1.3) on α, α', β and β' we shall need

$$\alpha < 2\beta, \qquad \alpha' < 2\beta'. \tag{4.5}$$

THEOREM 4.1. *Let the conditions* (1.3), (3.1), (4.2) *and* (4.5) *hold. Then*

(4.6) $$-2\varepsilon \log \lambda_\varepsilon \to J \quad as\ \varepsilon \to 0.$$

PROOF. If $\phi < \min(\hat{x},1-\hat{x})$, then by Corollary 3.3 we have

(4.7) $$\lambda_\varepsilon \leqslant \lambda_\varepsilon(\delta,1-\delta);$$

recall that $\lambda_\varepsilon(\delta,1-\delta)$ is the regular principal eigenvalue of the operator L_ε with zero Dirichlet data on the boundary of $[\delta,1-\delta]$. By hypothesis (4.2), $b(\delta)>0$, and $b(1-\delta)<0$. Since also $a(x)>0$ on $[\delta,1-\delta]$ we may apply the second part of Theorem 1.1 of [3] to conclude that

(4.8) $$\lim_{\varepsilon\to 0}\left[-2\varepsilon \log \lambda_\varepsilon(\delta,1-\delta)\right] \geqslant J_\delta.$$

We shall next show that

(4.9) $$\varliminf_{\delta\to 0} J_\delta \geqslant J.$$

By the definition of J_δ there exists an absolutely continuous function $\phi(t)$, $t\in[0,T_0]$, satisfying $\phi(0)=\hat{x}$, $\delta<\phi(t)<1-\delta$ if $0\leqslant t<T_0$, $\phi(T_0)=\delta$ or $\phi(T_0)=1-\delta$ and

(4.10) $$I_{T_0}(\phi) < J_\delta+\delta.$$

For definiteness suppose that $\phi(T_0)=\delta$. Extend ϕ to $T_0<t\leqslant T_0+1$ by

$$\phi(t)=\delta(T_0+1-t)^\mu, \qquad \mu>1,.$$

where μ will be determined shortly. Clearly the extended function ϕ is absolutely continuous. Since $\phi(T_0+1)=0$ we have

(4.11) $$J \leqslant I_{T_0+1}(\phi).$$

Now

$$I_{T_0+1}(\phi)-I_{T_0}(\phi) \leqslant 2\int_{T_0}^{T_0+1}\frac{(\phi'(t))^2}{a(\phi(t))}\,dt+2\int_{T_0}^{T_0+1}\frac{b(\phi(t))^2}{a(\phi(t))}.$$

Substituting $x=T_0+1-t$ we get

$$I_{T_0+1}(\phi)-I_{T_0}(\phi) \leqslant C\int_0^1\frac{\delta^2x^{2\mu-2}}{\delta^\alpha x^{\alpha\mu}}\,dx+C\int_0^1\frac{\delta^{2\beta}x^{2\beta\mu}}{\delta^\alpha x^{\alpha\mu}}\,dx$$

$$=C\delta^{2-\alpha}\int_0^1 x^{\mu(2-\alpha)-2}\,dx+C\delta^{2\beta-\alpha}\int_0^1 x^{\mu(2\beta-\alpha)}\,dx,$$

where C is independent of δ. Taking $\mu>1/(2-\alpha)$ we have

$$I_{T_0+1}(\phi)-I_{T_0}(\phi)\leqslant C\delta^\nu, \qquad \nu=\min(2-\alpha,2\beta-\alpha),$$

where we have taken C as a generic constant, a practice which we shall continue in the future. Using (4.10) and (4.11) in conjunction with this inequality gives

$$J_\delta > I_{T_0+1}-\delta-C\delta^\nu \geqslant J-\delta-C\delta^\nu.$$

Inequality (4.9) follows from this. From (4.7), (4.8) and (4.9) we get

$$\varliminf_{\varepsilon\to 0}\,[-2\varepsilon \log \lambda_\varepsilon] \geq J. \tag{4.12}$$

If we can prove the reverse inequality with $\varliminf$ replaced by $\varlimsup$ we shall have established Theorem 4.1. Toward this end let γ and η be positive numbers, to be determined later, with $0 < \eta < \gamma < \min(\hat{x}, 1 - \hat{x})$. Let $a_\eta(x) \in C^1(0,1)$ satisfying

$$a_\eta(x) = \begin{cases} \dfrac{a(x)}{x^\alpha}(x+\eta)^\alpha & \text{if } x \in (0,\gamma/2), \\ \dfrac{a(x)}{(1-x)^{\alpha'}}(1-x)+\eta^{\alpha'} & \text{if } x \in (1-\gamma/2,1), \\ a(x) & \text{if } x \in (\gamma, 1-\gamma), \end{cases} \tag{4.13}$$

and

$$|a_\eta(x) - a(x)| < C\eta/\gamma \quad \text{if } x \in (\gamma/2,\gamma) \text{ or } x \in (1-\gamma, 1-\gamma/2), \tag{4.14}$$

where C is independent of η and γ for $0 < \eta < \gamma$. That such a function can be constructed is easily seen. Indeed set $\tilde{a}(x) = a(x)(1 + \eta/x)^\alpha$ if $x \in (0,\gamma]$, $\tilde{a}(x) = a(x)(1 + \eta/(1-x))^{\alpha'}$ if $x \in [1-\gamma, 1)$. Let $\chi_1, \chi_2 \in C^\infty[0,1]$ such that $\chi_1 = 1$ for $x \in (0,\gamma/2)$ and $x \in (1-\gamma/2, 1)$, $\chi_1 = 0$ for $x \in (\gamma, 1-\gamma)$ and $\chi_1 + \chi_2 = 1$. Set $a_\eta = \chi_1\tilde{a} + \chi_2 a$, so that if $x \in (\gamma/2,\gamma)$, say, then

$$\begin{aligned} |a_\eta(x) - a(x)| &= a(x)\chi_1(x)|(1+\eta/x)^\alpha - 1| \\ &= \alpha a(x)\chi_1(x)(\eta/x)(1+\eta'/x)^{\alpha-1}, \qquad 0 < \eta' < \eta. \end{aligned}$$

If $\alpha \leq 1$, then $(1+\eta'/x)^{\alpha-1} \leq 1$, and if $\alpha > 1$, $(1+\eta'/x)^{\alpha-1} \leq (1 + 2\eta/\gamma)^{\alpha-1} < 3^{\alpha-1}$. The same type of considerations hold for $x \in (1-\gamma, 1-\gamma/2)$. Thus the existence of an a_η satisfying (4.13) and (4.14) is established.

Let $b_\eta \in C^1(0,1)$ satisfying

$$b_\eta(x) = \begin{cases} \dfrac{b(x)}{x^\beta}(x+\eta)^\beta & \text{if } x \in (0,\gamma/2), \\ \dfrac{b(x)}{(1-x)^{\beta'}}(1-x+\eta)^{\beta'} & \text{if } x \in (1-\gamma/2,1), \\ b(x) & \text{if } [\gamma, 1-\gamma], \end{cases} \tag{4.15}$$

and

$$|b_\eta(x) - b(x)| \leq C\eta/\gamma \quad \text{if } x \in (\gamma/2,\gamma) \text{ or } x \in (1-\gamma, 1-\gamma/2), \tag{4.16}$$

where C is independent of γ and η provided $0 < \eta < \gamma$. That such a function

b_η can be constructed follows the same reasoning as above.

Let us consider the eigenvalue problem

(4.17) $$L_{\eta\varepsilon} \equiv (\varepsilon/2)a_\eta u'' + b_\eta u' = -\lambda u, \quad u(0) = u(1) = 1.$$

Denote the principal eigenvalue for this problem by λ_ε^η. Since $L_{\eta\varepsilon}$ is nondegenerate in [0,1] and $b_\eta(0) > 0$ and $b_\eta(1) < 0$ we may apply the first part of Theorem 1.1 of [3] and conclude that

(4.18) $$\overline{\lim_{\varepsilon\to 0}}\,[-2\varepsilon \log \lambda_\varepsilon^\eta] \leqslant J^\eta,$$

where J^η is defined in the same way as J in (4.4) with the exception that $a(x)$ and $b(x)$ are replaced by $a_\eta(x)$ and $b_\eta(x)$, respectively.

Let

$$p_{\varepsilon\eta}(x) = \exp \int_0^x \frac{2b_\eta(t)}{\varepsilon a_\eta(t)}\,dt.$$

Then (4.17) is equivalent to

(4.19) $$(p_{\varepsilon\eta}u')' = -\lambda(2p_{\varepsilon\eta}/\varepsilon a_\eta)u, \quad u(0) = u(1) = 0.$$

Since the eigenvalue problem (4.19) is nondegenerate and selfadjoint the principal eigenvalue λ_ε^η is determined by the variational formula

(4.20) $$\lambda_\varepsilon^\eta = \inf_{w\neq 0}\left[\int_0^1 p_{\varepsilon\eta}(w')^2 \Big/ \int_0^1 \frac{2p_{\varepsilon\eta}w^2}{\varepsilon a_\eta}\right],$$

where $w(0) = w(1) = 0$.

Recall that we proved in §3 that the eigenvalue problem

$$(p_\varepsilon u')' = -\lambda(2p_\varepsilon/\varepsilon a)u, \quad u(0) = u(1) = 0,$$

has a solution $w_\varepsilon(x) > 0$ in (0,1), where

$$p_\varepsilon(x) = \exp \int_0^x \frac{2b}{\varepsilon a}.$$

We normalize w_ε so that

(4.21) $$\int_0^1 \frac{2p_\varepsilon w_\varepsilon^2}{\varepsilon a} = 1.$$

By Theorem 3.4 we have

(4.22) $$\lambda_\varepsilon = \int_0^1 p_\varepsilon(w_\varepsilon')^2.$$

If we use w_ε in the quotient on the right-hand side of (4.20) and use (4.21) we should get a comparison between λ_ε^η and λ_ε, at least for all sufficiently small η. However, in order to do this we need a comparison between the integrals involving the functions having an η subscript and the integrals involving the functions without an η subscript. We now proceed to develop this comparison.

If $0 \leq t \leq x \leq \gamma$, then

$$p_{\varepsilon\eta}(x) \leq K_\varepsilon p_\varepsilon(t) \tag{4.23}$$

holds if and only if

$$K_\varepsilon \geq \exp\left[\int_0^x \frac{2b_\eta}{\varepsilon a_\eta} - \int_0^t \frac{2b}{\varepsilon a}\right]. \tag{4.24}$$

Using (4.13) and (4.15) we find that

$$\exp\left[\int_0^{\gamma/2} \frac{2b_\eta}{\varepsilon a_\eta}\right] \leq C \exp \frac{C\gamma^{1+\beta-\alpha}}{\varepsilon};$$

$$\exp\left[\int_0^{\gamma/2} \frac{2b}{\varepsilon a}\right] \leq C \exp \frac{C\gamma^{1+\beta-\alpha}}{\varepsilon}.$$

Using (4.14), (4.16) and the fact that $a(x)$ and $b(x)$ are positive in $[\gamma/2,\gamma]$ we find

$$\exp\left[\int_{\gamma/2}^{\gamma} \frac{2b_\eta}{\varepsilon a_\eta}\right] \leq C \exp \frac{C\gamma^{1+\beta-\alpha}}{\varepsilon},$$

for all sufficiently small η depending on γ, say $\eta \leq \eta_0(\gamma)$. Also

$$\exp\left[\int_{\gamma/2}^{\gamma} \frac{2b}{\varepsilon a}\right] \leq C \exp \frac{C\gamma^{1+\beta-\alpha}}{\varepsilon}.$$

Using these estimates in (4.24) we conclude that (4.23) holds if

$$K_\varepsilon = C \exp(C\gamma^{1+\beta-\alpha}/\varepsilon), \qquad \eta \leq \eta_0(\gamma), \tag{4.25}$$

where C is a positive constant independent of η and γ.

Next, taking $0 < x < \gamma$ we have

$$(w_\varepsilon(x))^2 \leq \left(\int_0^x |w_\varepsilon'(t)|\, dt\right)^2.$$

Hence, by (4.23)

$$p_{\varepsilon\eta}(x)(w_\varepsilon(x))^2 \leq \left[p_{\varepsilon\eta}(x)^{1/2} \int_0^w |w_\varepsilon'(t)|\, dt\right]^2$$
$$\leq K_\varepsilon \left[\int_0^x p_\varepsilon(t)^{1/2} |w'(t)|\, dt\right]^2.$$

Applying the Cauchy-Schwarz inequality to the last integral and using (4.22) we get

$$p_{\varepsilon\eta}(x)(w_\varepsilon(x))^2 \leqslant K_\varepsilon x \int_0^x p_\varepsilon (w_\varepsilon')^2 \leqslant K_\varepsilon \lambda_\varepsilon x. \tag{4.26}$$

Using (4.12) and the value of K_ε from (4.25) we get

$$p_{\varepsilon\eta}(x)(w_\varepsilon(x))^2 \leqslant Ce^{-J/4\varepsilon}x, \qquad x \in (0,\gamma), \tag{4.27}$$

provided γ is chosen sufficiently small so that $C\gamma^{1+\beta-\alpha} < J/4$. Recall that $\alpha < 1 + \beta$ so that this can be accomplished. From this it follows that

$$\int_0^\gamma \frac{2p_{\varepsilon\eta}(x)(w_\varepsilon(x))^2}{\varepsilon a_\eta(x)}\, dx \leqslant \frac{C}{\varepsilon} e^{-J/4\varepsilon} \int_0^\gamma \frac{x}{x^\alpha}\, dx \leqslant Ce^{-c/\varepsilon}, \tag{4.28}$$

where C and c are positive constants independent of η,γ and ε. Inequality (4.28) holds provided γ is sufficiently small, but fixed, and $\eta \leqslant \eta_0(\gamma)$. Similarly one proves that the integrals

$$\int_{1-\gamma}^1 \frac{2p_{\varepsilon\eta}w_\varepsilon^2}{\varepsilon a_\eta}, \quad \int_0^\gamma \frac{2p_\varepsilon w_\varepsilon^2}{\varepsilon a}, \quad \int_{1-\gamma}^1 \frac{2p_\varepsilon w_\varepsilon^2}{\varepsilon a}$$

are bounded by the right-hand side of (4.28). Recalling the normalization (4.21) and the fact that

$$p_{\varepsilon\eta}(x) = p_\varepsilon(x), \quad a_\eta(x) = a(x) \quad \text{for } x \in (\gamma, 1-\gamma),$$

we conclude that

$$\left| 1 - \int_0^1 \frac{2p_{\varepsilon\eta}w_\varepsilon^2}{\varepsilon a_\eta} \right| \leqslant Ce^{-c/\varepsilon}. \tag{4.29}$$

Taking $w = w_\varepsilon$ in (4.20) we get from (4.29)

$$\lambda_\varepsilon^\eta \leqslant (1 + Ce^{-c/\varepsilon}) \int_0^1 p_{\varepsilon\eta}(w_\varepsilon')^2 \tag{4.30}$$

for all sufficiently small ε, independent of η and γ. From (4.23) and (4.25) we get

$$\int_0^\gamma p_{\varepsilon\eta}(w_\varepsilon')^2 \leqslant C \exp\left(\frac{C\gamma^{1+\beta-\alpha}}{\varepsilon} \right) \int_0^\gamma p_\varepsilon (w_\varepsilon')^2.$$

A similar inequality holds when the integrals are taken over $[1-\gamma, 1]$. Also, since $p_{\varepsilon\eta}(x) = p_\varepsilon(x)$ if $x \in [\gamma, 1-\gamma]$ we get from (4.22) and (4.30)

$$\lambda_\varepsilon^\eta \leqslant \lambda_\varepsilon(1 + Ce^{-c/\varepsilon})C \exp(C\gamma^{1+\beta-\alpha}/\varepsilon). \tag{4.31}$$

Using (4.18) we conclude from (4.31) that

$$\overline{\lim_{\varepsilon\to 0}}\, [-2\varepsilon \log \lambda_\varepsilon] \leqslant C\gamma^{1+\beta-\alpha} + J^\eta. \tag{4.32}$$

This inequality holds for all $\gamma \leqslant \gamma_0$ and $\eta \leqslant \eta_0(\gamma)$, and C is a constant independent of γ and η.

By the definition of J there exists an absolutely continuous function $\phi(t)$, $t \in [0,T]$, with $\phi(0) = \hat{x}$ and $\phi(T) = 0$, or $\phi(T) = 1$ such that

$$I_T(\phi) < J + \gamma. \tag{4.33}$$

Let T_0 be the smallest t for which either $\phi(t) = \gamma$ or $\phi(t) = 1 - \gamma$. Suppose for definiteness that $\phi(T_0) = \gamma$. Let us set

$$I_{T_0}^{\eta} = \int_0^{T_0} \frac{[\phi'(t) - b_\eta(\phi(t))]^2}{a_\eta(\phi(t))}\, dt. \tag{4.34}$$

Extend ϕ to $(T_0, T_0 + 1]$ by

$$\phi(t) = \gamma(T_0 + 1 - t)^\mu, \qquad \mu > \max(1, 1/(2 - \alpha)).$$

The extended function is absolutely continuous and

$$\int_{T_0}^{T_0+1} \frac{[\phi'(t) - b_\eta(\phi(t))]^2}{a_\eta(\phi(t))}\, dt \le 2\int_{T_0}^{T_0+1} \frac{(\phi'(t))^2}{a_\eta(\phi(t))} + 2\int_{T_0}^{T_0+1} \frac{(b_\eta(\phi(t)))^2}{a_\eta(\phi(t))}\, dt.$$

By computations similar to those which we have made between (4.11) and (4.12) we obtain an upper bound for the right-hand side which is $C\gamma^\nu$, $\nu = \min(2 - \alpha, 2\beta - \alpha)$, provided η is sufficiently small, say $\eta < \eta_1(\gamma)$. Consequently, recalling the definition of $I_{T_0}^{\eta}$ from (4.34) we have

$$\int_0^{T_0+1} \frac{[\phi'(t) - b_\eta(\phi(t))]^2}{a_\eta(\phi(t))}\, dt < I_{T_0}^{\eta} + C\gamma^\nu. \tag{4.35}$$

Since $\phi(0) = \hat{x}$, and $\phi(T_0 + 1) = 0$, J^η is smaller than or equal to the left-hand side of (4.35). Hence, using (4.33) we get

$$J^\eta < I_{T_0}^{\eta} + C\gamma^\nu < J + \gamma + C\gamma^\nu.$$

Using this inequality in (4.32) and noting that γ can be arbitrarily small we get

$$\overline{\lim_{\varepsilon \to 0}}\, [-2\varepsilon \log \lambda_\varepsilon] < J.$$

The proof of Theorem 4.1 is complete.

Let us now complement Theorem 4.1 by explicitly evaluating the constant J.

THEOREM 4.2. *The constant J in Theorem* 4.1 *is given by*

$$J = \min\left\{ \int_0^{\hat{x}} \frac{4b(t)}{a(t)}\, dt, \int_1^{\hat{x}} \frac{4b(t)}{a(t)}\, dt \right\}. \tag{4.36}$$

PROOF. Let us set

$$J_0(x) = \inf\{I_T(\phi): \phi \in \Phi(x,T;0,1), \phi(T) = 0\},$$
$$J_1(x) = \inf\{I_T(\phi): \phi \in \Phi(x,T;0,1), \phi(T) = 1\},$$
$$J(x) = \inf\{I_T(\phi): \phi \in \Phi(x,T;0,1)\}.$$

Clearly, these functionals do not change if we restrict the competing functions $\phi(t)$ to be monotone, and it is immediate that

$$J(x) = \min\{J_0(x), J_1(x)\}.$$

Further, it is clear that $J_0(x)$ is a nondecreasing function of x, and $J_1(x)$ is a nonincreasing function of x. It is also easy to see that $J_0(x), J_1(x)$ and $J(x)$ are Lipschitz continuous in $(0,1)$.

Let $E_0 = \{x: J_0(x) < J_1(x)\}$ and $E_1 = \{x: J_1(x) < J_0(x)\}$. E_0 and E_1 are intervals and upon setting

$$x_0 = \sup\{x: x \in E_0\}, \quad x_1 = \inf\{x: x \in E_1\}$$

it follows that $x_0 \leqslant x_1$. We claim that

$$J_0(x) = J(x) = J_1(x) = \text{const for } x \in [x_0, x_1]. \tag{4.37}$$

Indeed if $x > x_1$, then $J(x) = J_1(x)$, and taking $x \downarrow x_1$ and using the continuity of these functions, we get $J(x_1) = J_1(x_1)$. Next, if $x < x_1$, by the definition of x_1 it follows that $J_0(x) \leqslant J_1(x)$ so that $J_0(x) = J(x)$. By letting $x \uparrow x_1$ we get $J_0(x_1) = J(x_1)$ so that we have

$$J_0(x_1) = J(x_1) = J_1(x_1). \tag{4.38}$$

Similarly we arrive at the fact that

$$J_0(x_0) = J(x_0) = J_1(x_0). \tag{4.39}$$

By the monotonicity of J_0 and J_1 we have

$$\begin{aligned} J_0(x_0) &\leqslant J_0(x) \leqslant J_0(x_1), \\ J_1(x_0) &\geqslant J_1(x) \geqslant J_1(x_1), \end{aligned} \qquad x \in [x_0, x_1].$$

Hence,

$$\min\{J_0(x_0), J_1(x_1)\} \leqslant J(x) \leqslant \min\{J_0(x_1), J_1(x_0)\}.$$

Thus from (4.38) and (4.39) it follows that for $x \in [x_0, x_1]$,

$$J(x) = \text{constant} = J_j(x_k), \qquad 0 \leqslant j,k \leqslant 1.$$

Using the monotonicity of $J_j(x)$ we get (4.37).

Let $x > \max\{x_0, \hat{x}\}$ so that $J(x) = J_1(x)$. For $h > 0$ so that $x + h < 1$, upon setting

$$\Phi_h(x,T) = \{\phi: \phi \in \Phi(x,T;0,x+h), \phi(T) = x + h\},$$

it is easily established that

$$J(x) = \inf\{I_T(\phi): \phi \in \Phi_h(x,T)\} + J(x+h).$$

Hence

$$-\frac{J(x+h)-J(x)}{h} = \frac{1}{h}\inf\{I_T(\phi): \phi \in \Phi_h(x,T)\}. \tag{4.40}$$

If we choose $\phi(t) = x + t$, $T = h$, we see that

$$\inf\{I_T(\phi): \phi \in \Phi_h(x,T)\} \leqslant Ch.$$

Thus for a fixed constant C_0,

$$\inf\{I_T(\phi): \phi \in \Phi_h(x,T)\} = \inf\{I_T(\phi): \phi \in \Phi_h(x,T), I_T(\phi) \leqslant C_0 h\}. \tag{4.41}$$

Take any ϕ in the right-hand side of (4.41) and write

$$\dot{\phi} - b(\phi) = f, \tag{4.42}$$

so that $\int_0^T f^2 \leqslant Ch$. Let

$$\dot{\psi} - b(\psi) = 0, \qquad \psi(0) = x. \tag{4.43}$$

Subtracting (4.43) from (4.42), multiplying the result by $\phi - \psi$ and integrating over $[0,t]$ we get

$$(\phi - \psi)^2(t) \leqslant C\int_0^t (\phi - \psi)^2 + \int_0^t f^2 \leqslant C\int_0^t (\phi - \psi)^2 + Ch.$$

Thus by Gronwall's inequality

$$(\phi - \psi)^2(t) \leqslant Ce^{Ct}h. \tag{4.44}$$

Suppose that $T \geqslant h^{\theta/2}$ for some θ, $0 < \theta < 1$. Then taking $t = h^{\theta/2}$ and noting that since $x > \hat{x}$, $\psi(t) \leqslant x - Ch^{\theta/2}$, $C > 0$, and $\phi(t) \geqslant x$,

$$Ch^\theta \leqslant [\phi(t) - \psi(t)]^2 \leqslant C\exp(Ch^{\theta/2})h.$$

Clearly, this is impossible for all sufficiently small h. Thus for every θ, $0 < \theta < 1$, $T < h^{\theta/2}$, and we may write

$$\begin{aligned} &\inf\{I_T(\phi): \phi \in \Phi_h(x,T)\} \\ &\qquad = \inf\{I_T(\phi): \phi \in \Phi_h(x,T), I_T(\phi) \leqslant C_0 h, T < h^{\theta/2}\}. \end{aligned} \tag{4.45}$$

Suppose now that ϕ is an element of the right-hand side of (4.45). Then we have $a(x) - a(\phi(t)) = O(h)$ and $b(x) - b(\phi(t)) = O(h)$, where $O(h)$ is independent of ϕ. Thus for all sufficiently small h,

$$\begin{aligned} I_T(\phi) = &\int_0^T \frac{[\dot{\phi}(t) - b(x)]^2}{a(x)}\, dt(1 + O(h)) \\ &+ 2\int_0^T \frac{[\dot{\phi}(t) - b(x)][b(x) - b(\phi(t))]}{a(\phi(t))}\, dt \\ &+ \int_0^T \frac{[b(x) - b(\phi(t))]^2}{a(\phi(t))}\, dt. \end{aligned}$$

The last two terms on the right may be estimated by

$$2\int_0^T \frac{[\dot{\phi}(t)-b(\phi(t))][b(x)-b(\phi(t))]}{a(\phi(t))}\,dt-\int_0^T \frac{[b(x)-b(\phi(t))]^2}{a(\phi(t))}\,dt$$

$$< 2I_T(\phi)^{1/2}\left\{\int_0^T \frac{[b(x)-b(\phi(t))]^2}{a(\phi(t))}\,dt\right\}^{1/2} = O(h^{3/2+\theta/4}),$$

where $O(h^{3/2+\theta/4})$ is independent of ϕ. Consequently,

$$(4.46)\quad \frac{1}{h}\inf I_T(\phi) = \frac{1}{h}\inf \int_0^T \frac{[\dot{\phi}(t)-b(x)]^2}{a(x)}(1+O(h)) + O(h^{1/2+\theta/4}).$$

For fixed T, it follows by the standard convexity argument in a Hilbert space that the functional $h^{-1}\int_0^T[\dot{\phi}(t)-b(x)]^2/a(x)\,dt$ takes on a minimum as ϕ ranges over all absolutely continuous functions on $[0, T]$ with $\phi(0)=x$, $\phi(T)=x+h$. It may then be verified by using Euler's equation that the minimum is taken on when $\dot{\phi}(t)=\lambda=$ constant; i.e., $\phi(t)=x+\lambda t$. Since $\phi(T)=x+h$ we must have $T=h/\lambda$. Thus, to find the minimum as T varies it is enough to minimize

$$F(\lambda)=\frac{1}{h}\int_0^{h/\lambda}\frac{[\lambda-b(x)]^2}{a(x)}\,dt=\frac{[\lambda-b(x)]^2}{a(x)\lambda}$$

over all $\lambda > 0$. A calculation shows that the minimum is at $\lambda = -b(x)$, and since $x > \hat{x}$, $b(x) < 0$ so that $\lambda > 0$. The minimum is $-4b(x)/a(x)$. From (4.46) it follows that

$$\lim_{h\to 0}\frac{1}{h}\inf I_T(\phi) = -\frac{4b(x)}{a(x)}.$$

From (4.40) we conclude that

$$(4.47)\qquad J'(x) = 4b(x)/a(x),$$

where $J'(x)$ is the right derivative. Similar arguments show that (4.47) is also valid for left derivatives and also when $0 < x < \min(\hat{x},x_1)$.

We now have three cases to distinguish:

(i) $x_0 < x_1 < \hat{x}$. By integrating (4.47) over $[x,y] \subset (\hat{x},1)$, letting $x\downarrow\hat{x}$, $y\uparrow 1$, using the fact that $J(y)\to 0$ as $y\to 1$ and the continuity of J at $\hat{x}$ we get

$$J(\hat{x}) = \int_1^{\hat{x}}\frac{4b}{a}.$$

(ii) $\hat{x} < x_0 < x_1$. Again, by integrating over $[x,y] \subset (0,\hat{x})$, letting $x\downarrow 0$ and

$y\uparrow\hat{x}$, using the fact that $J(x)\to 0$ as $x\to 0$ and the continuity of J at $\hat{x}$ we have $J(\hat{x}) = \int_0^{\hat{x}} 4b/a$.

(iii) $x_0 < \hat{x} < x_1$. This case can be ruled out. Indeed, if $\hat{x} < x < x_1$ then by (4.47) we have $J'(x) = 4b(x)/a(x) \neq 0$ which contradicts (4.37).

Now, in case (i) we have

$$\int_0^{\hat{x}} \frac{4b}{a} > \int_0^{x_1} \frac{4b}{a} = J(x_1) = J_1(x_1) > J_1(\hat{x}) = J(\hat{x}) = \int_1^{\hat{x}} \frac{4b}{a}.$$

In case (ii) we have

$$\int_1^{\hat{x}} \frac{4b}{a} > \int_1^{x_0} \frac{4b}{a} = J(x_0) = J_0(x_0) > J_0(\hat{x}) = J(\hat{x}) = \int_0^{\hat{x}} \frac{4b}{a}.$$

Hence, in either case we have (4.36), and the theorem is proved.

REMARK. Most of the results of this paper extend to "essentially" selfadjoint elliptic operators such as

$$L_\varepsilon u \equiv \varepsilon x(1-x)u_{xx} + \varepsilon y(1-y)u_{yy} + x(1-x)(A-x-\alpha y)u_x$$
$$+y(1-y)(A-y-\alpha x)u_y \qquad (0 < A < 1+\alpha < 2)$$

in a square $Q = (0,1) \times (0,1)$, where $\varepsilon > 0$. This is a selfadjoint operator with respect to the measure

$$\frac{\exp\{A(x+y) - (x^2+y^2) - \alpha xy\}}{x(1-x)y(1-y)}\, dx\, dy.$$

The point $(\hat{x},\hat{y}) = (A/(1+\alpha), A/(1+\alpha))$ is a stable equilibrium point for

$$dx/dt = b_1(x,y), \quad dy/dt = b_2(x,y),$$

where

$$b_1(x,y) = x(1-x)(A-x-\alpha y), \quad b_2(x,y) = y(1-y)(A-y-\alpha x).$$

There exists a "principal" eigensolution $(u_\varepsilon,\lambda_\varepsilon)$ in the following sense:

$$Lu_\varepsilon = -\lambda_\varepsilon u_\varepsilon \quad \text{in } Q,$$

$u_\varepsilon = 0$ on ∂Q, in some generalized sense,

$$\lambda_\varepsilon = \int_Q\int \left[\frac{1}{x(1-x)} \left(\frac{\partial u_\varepsilon}{\partial x} \right)^2 + \frac{1}{y(1-y)} \left(\frac{\partial u_\varepsilon}{\partial y} \right)^2 \right] dx\, dy,$$

$u_\varepsilon > 0$ in Q;

further, if (v,μ) is any other eigensolution, then $\mu > \lambda_\varepsilon$.

Proceeding by the method of §4 one can prove that

$$-2\varepsilon \log \lambda_\varepsilon \to J \quad \text{as } \varepsilon \to 0$$

where

$$J = \inf_{\phi} \int_0^T \left[\frac{(\dot{\phi}_1 - b_1(\phi))^2}{\phi_1(1-\phi_1)} + \frac{(\dot{\phi}_2 - b_2(\phi))^2}{\phi_2(1-\phi_2)} \right] dt$$

and ϕ varies over the class of absolutely continuous functions defined on $[0,T]$ such that $\phi(0) = (\hat{x},\hat{y})$, $\phi(t) \in Q$ for $t \in [0,T]$ and $\phi(T) \in \partial Q$; T is arbitrary.

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